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PATH INDEPENDENT INTEGRALS FOR CYLINDRICAL SHELLS AND SHELLS OF REVOLUTION

K. K. Lo
Division of Engineering, Brown University, Providence, RI 02912 U.S.A.

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Abstract—In this paper we examine the path-independence of certain integrals in circular cylindrical shells and shells of revolution. It is shown that a circular cylindrical shell has at least two conservation laws associated with its translational and rotational invariance while for a shell of revolution, we find that there is a conservation law derived from invariance to rotation. Next, we relate these integrals to energy release rates associated with translation and rotation of a cavity. An example is then considered and extension of these path-independent integrals to nonlinear membrane theory is carried out.

NOTATION

$b_{\alpha\beta}$ second fundamental (curvature) tensor $= N^i{}_{,\alpha} x^i{}_{,\beta}$
$g_{\alpha\beta}$ metric tensor of the undeformed shell $= x^i{}_{,\alpha} x^i{}_{,\beta}$
$\bar{g}_{\alpha\beta}$ metric tensor of the deformed shell
ds arc length element
dA surface area element
N^i unit normal to surface
$x^i{}_{,\alpha}$ surface base vectors
R radius of the cylinder
h thickness of the shell
λ $= [3/4(1-\nu^2)]^{1/4} c/\sqrt{(Rh)}$

INTRODUCTION

Conservation laws (or path-independent integrals) for linear and non-linear elastic materials have been considered by various authors[1–4]. One of these integrals, the J integral, has since been applied extensively to fracture mechanics with much success. In this paper, we examine similar type of integrals for cylindrical and axisymmetric shells in the context of thin shell theory obeying Kirchoff's hypothesis. Path-independent integrals in shells have been considered by Bergez and Radenkovic[5] and Bergez[6]. However, it appears that they have not placed any restrictions on the geometry of the shells and based on the considerations on invariance in this study, such integrals are not path independent in general. Studies made by Knowles and Sternberg[2] show that conservation laws are derived from the invariance of a variational principle to a group of transformations which corresponds to translation, rotation and expansion. One would not, therefore, expect path-independent integrals to exist in general for shells except those which enjoy a high degree of symmetry.

In this paper, it is shown that J and related integrals, generalized in an appropriate manner, are path independent for circular cylindrical and axisymmetric shells in the context of linear elastic shell theory as well as nonlinear membrane theory. These integrals are also related to energy release rates associated with translation and rotation. Finally, one of the integrals is applied to an example to illustrate its application.

Lines of curvature coordinates ($\xi^1 = z$, $\xi^2 = \theta$) are implied throughout the paper although general tensor forms will be used for simplicity. The summation convention will apply whenever repeated indexes are indicated with Greek indexes having range 2 and Latin indexes, 3. Contravariant and covariant components of a surface tensor will be denoted by Greek superscripts and subscripts, respectively. Commas will denote covariant differentiation of a tensor or ordinary differentiation of a scalar. In the case of a non-linear membrane theory, covariant differentiation will be based on the geometry of the undeformed shell. 3D Cartesian components are denoted by Latin indexes.

LINEAR SHELL THEORY

The starting point of the study is the linear elastic shell theory of Budiansky and Sanders[7] and Koiter[8]. Only the essentials of the theory are given here and details can be found in [7]. The displacement vector U^i of a material point in the middle surface of the shell is given by

$$U^i = u^\alpha x^i_{,\alpha} + wN^i \tag{1}$$

where u_α, w are the surface and surface-normal components of the displacement vector. The membrane and bending strain measures $E_{\alpha\beta}$, $K_{\alpha\beta}$ are given in terms of these components u_α, w by

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + b_{\alpha\beta}w \tag{2}$$

$$K_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) + \frac{1}{2}(b_\alpha{}^\gamma \omega_{\gamma\beta} + b_\beta{}^\gamma \omega_{\gamma\alpha}) \tag{3}$$

where the rotation $\phi_\alpha = -w_{,\alpha} + b_\alpha{}^\gamma u_\gamma$ and the rotation-about-the-normal tensor $\omega_{\alpha\beta} = (1/2)(u_{\alpha,\beta} - u_{\beta,\alpha})$. The conjugate stress variables are the symmetrical modified membrane tensor $N^{\alpha\beta}$ and bending moment $M^{\alpha\beta}$ which are adopted here for reasons enunciated in [7]. In the absence of surface loads, they satisfy the following equilibrium equations

$$\begin{aligned} &N^{\alpha\beta}_{,\alpha} + b_\gamma{}^\beta M^{\alpha\gamma}_{,\alpha} + \frac{1}{2}(b_\gamma{}^\beta M^{\alpha\gamma} - b_\gamma{}^\alpha M^{\gamma\beta})_{,\alpha} = 0 \\ &M^{\alpha\beta}_{,\alpha\beta} - b_{\alpha\beta}N^{\alpha\beta} = 0. \end{aligned} \tag{4}$$

As shown in [7], there are several stress measures which one can adopt. The above tensors $(M^{\alpha\beta}, N^{\alpha\beta})$ are chosen mainly because they satisfy the Principle of Virtual Work exactly when deformations are assumed to satisfy the Kirchoff's hypothesis, and they yield an exact static-geometric analogy. These properties shared by the stress-resultant $N^{\alpha\beta}$ and moment tensor $M^{\alpha\beta}$ are not limited to cylindrical shells and they remain valid in general linear elastic shell theory.

PATH-INDEPENDENT INTEGRALS

Consider deformations with displacements u_α, w on the middle surface of a circular cylindrical shell. Let S be a simply connected region on the middle surface boundary by a smooth closed curve C. The integrals considered in this paper are

$$J_\lambda = \oint_C \left\{ Wn_\lambda - \left[\bar{T}^\alpha \frac{\partial u_\alpha}{\partial \xi^\lambda} + M_n \psi_{(n),\lambda} + \bar{Q}\frac{\partial w}{\partial \xi^\lambda} \right] \right\} \mathrm{d}s \tag{5}$$

where $\lambda = 1$ and 2 refer respectively to the axial and circumferential co-ordinates. The strain energy density per unit area is denoted by W and on the contour C, the traction terms are

$$\bar{T}^\alpha = \left[N^{\alpha\beta} + \frac{1}{2}(b_\gamma{}^\alpha M^{\gamma\beta} - b_\gamma{}^\beta M^{\alpha\gamma}) + b_\omega{}^\alpha t^\omega t_\gamma M^{\gamma\beta} \right] n_\beta \tag{6}$$

$$\bar{Q} = M^{\alpha\beta}_{,\alpha} n_\beta + \frac{\partial}{\partial s}(M^{\alpha\beta} n_\alpha t_\beta), \qquad \psi_{(n),\lambda} = \frac{\partial \phi^\alpha}{\partial \xi^\lambda} n_\alpha \tag{7}$$

$$M_n = M^{\alpha\beta} n_\alpha n_\beta, \qquad M_t = -M^{\alpha\beta} n_\alpha t_\beta. \tag{8}$$

The term $\bar{T}^\alpha$ is the effective membrane force/length, $\bar{Q}$ is the effective transverse force/length, M_n is the normal bending moment, M_t the twisting moment and t_α is the unit tangent vector while n_α is the unit normal (in S) to C. For a circular cylinder, the integral J_λ in eqn (5) for $\lambda = 1$ and 2 and vanishes for all closed smooth paths C when the energy density W for a Hookean material is a quadratic function of the strains $E_{\alpha\beta}$, $K_{\alpha\beta}$

$$W = \frac{1}{2}\left[\frac{Eh}{1-\nu^2}\right][(1-\nu)E_{\alpha\beta}E^{\alpha\beta} + \nu E_\alpha{}^\alpha E_\beta{}^\beta] + \frac{1}{24}\left[\frac{Eh^3}{1-\nu^2}\right][(1-\nu)K_{\alpha\beta}K^{\alpha\beta} + \nu K_\gamma{}^\gamma K_\beta{}^\beta]. \tag{9}$$

Proof of the path independence of the integral J_λ is straightforward and the details are shown in Appendix 1. The result follows from the use of the following Divergence Theorem:

$$\oint_C W n_\lambda \, \mathrm{d}s = \int_S W_{,\lambda} \, \mathrm{d}A \tag{10}$$

when holds for a circular cylinder in lines-of-curvature coordinates in which $\sqrt{(g)} = \det(g_{\alpha\beta}) = R$, the radius of the cylinder, is a constant. Use is also made of the following property for a circular cylinder, satisfied also by a spherical surface,

$$b_{\alpha\beta,\lambda} = 0 \tag{11}$$

i.e. the covariant derivative of the second fundamental tensor vanishes for all coordinate systems. The conservation law (5) actually follows directly from the Principle of Virtual Work if one considers $(\partial u_\alpha/\partial \xi^\lambda, \partial w/\partial \xi^\lambda)$ as an admissible set of displacements for fixed λ.

ENERGY RELEASE RATES

While the proof of the path independence of the integrals in (5) is straightforward, it does not afford any particular physical insight in the interpretation of these conservation laws. In this section we relate the integrals in (5) to energy release rates associated with cavity translation and rotation, thus identifying them with the corresponding conservation laws in elasticity[2, 3].

From the paper by Budiansky and Rice[3], one has the expression for the energy release rate G of a 3D body as

$$G\delta t = \int_S W\mathbf{n} \cdot \delta\mathbf{x} \, \mathrm{d}A \tag{12}$$

where W is the strain energy per unit volume, $\mathbf{n}$ is the unit normal to the cavity S, $\delta\mathbf{x}$ is a virtual displacement of a portion of the boundary S and t is a time-like parameter. By analogy, on the middle surface of a two-dimensional thin shell, we have the energy release rate related to a line integral

$$G\delta t = \int_\Gamma W\mathbf{n} \cdot \delta\mathbf{x} \, \mathrm{d}s \tag{13}$$

where now W is the strain energy per unit area of the middle surface and Γ is the portion of the cavity with nonzero $\delta\mathbf{x}$. It is clear then the integral J_1 ($\lambda = 1$) in eqn (5) for a circular cylindrical shell corresponds to the energy release rate of a cavity translating in the axial direction with $\delta\mathbf{x}$ as any constant vector in that direction. One expects the conservation law J_1 to hold because of the translational invariance of the cylinder. However, it is not obvious that J_2 ($\lambda = 2$) in eqn (5) in the circumferential direction is a generalization of the J integral in the plane and hence that it should be path independent at all. On the other hand, we do expect that there is rotational invariance about the axis of the cylinder. Here the path independence of J_2 is rationalized by relating the integral to energy release rate associated with the rotation of a cavity about the cylinder axis.

A rotation about the cylinder axis can be written as

$$\delta\mathbf{x} = (\Omega \boldsymbol{\epsilon}_1) \times (R\mathbf{N}) \tag{14}$$

where Ω is the magnitude of the rotation vector (arbitrary) and $\mathbf{N}$ is a unit outer normal to the cylindrical surface, $\boldsymbol{\epsilon}_1$ is a base (unit) vector in the z direction. Consequently, $\delta\mathbf{x}$ is in the $\boldsymbol{\epsilon}_2$ (circumferential) direction and the expression for the energy release rate in (13) may be written as

$$G\delta t = (-\Omega R) \int_\Gamma W n_2 \, \mathrm{d}s. \tag{15}$$

If we now apply the Divergence Theorem (10) to eqn (15), we have

$$G\delta t = (-\Omega R)J_2 \tag{16}$$

and J_2 can be evaluated on any closed contour C surrounding the cavity. We see, therefore, that J_2 for the cylinder has the same interpretation as the L integral [2, 3] in elasticity as the energy release rate associated with cavity rotation about the z axis.

By relating the integral J_2 to the energy release rate associated with rotation, one would expect that path independence of J_2 should continue to hold for shells of revolution. That this is indeed the case can be shown by examining eqn (10) and the curvature tensor $b_\beta{}^\alpha$ which, in lines-of-curvature coordinates of a shell of revolution, is a function of the axial coordinate $\xi^1 = z$ only. For axisymmetric surfaces, the Divergence Theorem (10) still holds for $\lambda = 2$ because $g = \det(g_{\alpha\beta})$ is also a function of z only. Therefore, we have $\partial b_\beta{}^\alpha/\partial\xi^2 = 0$ so that the same proof applied to a cylindrical shell goes through for this case as well. However, we do not anticipate the conservation law J_1 to remain valid as there is no translational invariance in the axial direction for an arbitrary axisymmetric surface.

AN EXAMPLE

To illustrate the use of the conservation law J_2 in eqn (5), we consider an example in which we estimate the crack opening displacement of a circumferential crack in a circular cylindrical shell under axial tension with a Dugdale type plastic zone. Application of J_1 was demonstrated by Amazigo [9], who made use of the path independence implicitly. We assume the length of the crack to be $2c$ and a line plastic zone of length p ahead of the crack tip. We make use of the results of Duncan–Fama and Sanders [10] in which the aforementioned geometry under membrane loading is considered on the basis of shallow shell theory of Marguerre. Hence the results are only valid for short cracks ($c/R < 1$). Further, we assume a small scale yielding situation ($p/c \ll 1$) in which the membrane stress and bending moment singularities govern the far field behavior of a semi-infinite crack so that the effects of curvature enter only through the strengths of these singularities S, B (see eqn (17) below). Let the prescribed axial membrane stress be N_∞. The asymptotic field near the crack tip of a finite-length circumferential crack is [10, 11].

$$\begin{aligned} M_{zz} &\sim [12(1-\nu^2)]^{-1/2} K' h^2 B/\sqrt{(2\pi r)} \\ N_{zz} &\sim hK'\, S/\sqrt{(2\pi r)} \qquad \text{as } r \to 0,\ \theta = 0 \end{aligned} \tag{17}$$

where $K' = \sqrt{(\pi c)}(N_\infty/h)$ and r, θ are the polar coordinates with the crack tip as the origin and $\theta = 0$ is the extension of the crack axis. Using eqn (5) and the asymptotic fields in (17), we can calculate J_2 as in [9],

$$J_2 = \frac{hK'^2}{E}[S^2 + a_2 B^2] \tag{18}$$

where E is Young's modulus and a_2 is a constant of $\sigma(1)$ which depends on Poisson's ratio. From [10], the contribution from the strength of the bending singularity B is relatively small compared to the corresponding membrane stress term S ($B/S \simeq 0.1$). Consequently as a first approximation, we neglect the bending term in (18) and set J_2 as

$$J_2 \simeq hK'^2 S^2/E. \tag{19}$$

The term S is known [10] as a function of the curvature parameter $\lambda = [(3/4)(1-\nu^2)]^{1/4}(c/\sqrt{(Rh)})$. We now collapse the contour onto the line plastic zone, so that the only contribution is from the contour enclosing the strip yield zone. Since in general bending as well as stretching are involved, one should adopt a yield criterion as in [12] throughout the plastic zone,

$$N/h\sigma_y + 6|M/h^2\sigma_y| = 1 \tag{20}$$

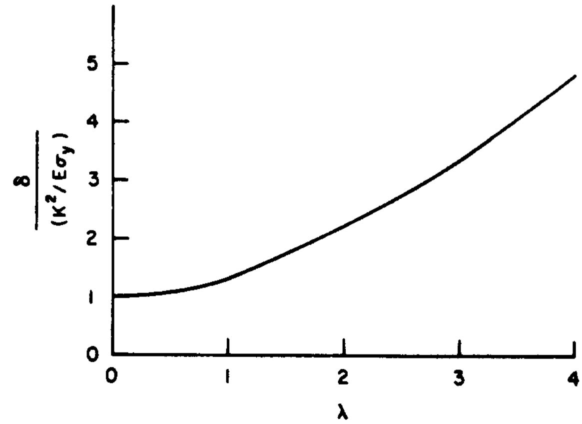

Fig. 1. Crack opening displacement.

where σ_y is the yield stress of the material. However, we note from [12] that the bending term $6|M/h|$ for a *circumferential* crack is again small compared to stretching N. As a first approximation and consistent with the assumption we made earlier for the far field, we assume that throughout the plastic zone, there is only the constant membrane stress-resultant $N = h\sigma_y$. The integral can then be easily evaluated to give

$$J_2 \simeq N\delta \tag{21}$$

where δ is the crack opening displacement. Making use of the path independence of J_2, we equate eqns (19) and (21) to give

$$\delta \simeq K'^2 S^2/E\sigma_y \tag{22}$$

where the strength of the membrane stress singularity S is given in [10] as a function of the curvature parameter λ. This gives a first order correction of the crack opening displacement due to curvature as shown in Fig. 1.

NONLINEAR MEMBRANE THEORY

It is found that similar conservation laws exist for a nonlinear membrane circular cylindrical shell, although the practical applicability in this case is doubtful. We restate here briefly the main ingredients for a nonlinear membrane theory. Details can be found in [13]. The *finite* strain measure $\hat{E}_{\alpha\beta}$ given by

$$\hat{E}_{\alpha\beta} = E_{\alpha\beta} + \frac{1}{2}(d^{\gamma}_{.\alpha} d_{\gamma\beta} + \phi_\alpha \phi_\beta) \tag{23}$$

where the linear part of the stretching strain $E_{\alpha\beta}$ is given in (2), ϕ_α is the same rotation as before and

$$d_{\alpha\beta} = u_{\alpha,\beta} + b_{\alpha\beta} w. \tag{24}$$

Along any curve C with normal n_α (in S) in the undeformed state, the edge force per unit undeformed length Q^i is

$$Q^i = T^\alpha x^i_{,\alpha} + QN^i \tag{25}$$

where

$$\begin{aligned} T_\alpha &= (g_{\alpha\beta} + d_{\alpha\beta}) n^{\gamma\beta} n_\gamma \\ Q &= -\phi_\beta n^{\alpha\beta} n_\alpha \end{aligned} \tag{26}$$

The Kirchoff stress-resultant $n^{\alpha\beta}$ is related to the membrane stress-resultant by

$$n^{\alpha\beta} = \sqrt{(\bar{g}/g)}\, N^{\alpha\beta} \tag{27}$$

where $\bar{g} \equiv \det(\bar{g}_{\alpha\beta})$. The path-independent integrals are then given by

$$J_\lambda = \int_C [Wn_\lambda - T^\gamma u_{\gamma,\lambda} - Qw_{,\lambda}]\, \mathrm{d}s \tag{28}$$

where now W is the strain energy per unit undeformed area of the middle surface such that it is related to the Kirchoff stress resultant by

$$n^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial W}{\partial E_{\alpha\beta}} + \frac{\partial W}{\partial E_{\beta\alpha}}\right). \tag{29}$$

The proof of the conservation law (28) is given in Appendix 2.

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APPENDIX 1

Proof of path independence of J_λ for a cylindrical shell

To prove the pathe independence of J_λ, we apply the Divergence Theorem in (10) and eqns (6–9) to (5) to obtain the surface integral over S bounded by, C,

$$\begin{aligned} J_\lambda = \int_S \Big\{ & M^{\alpha\beta}\frac{\partial K_{\alpha\beta}}{\partial \xi^\lambda} + N^{\alpha\beta}\frac{\partial E_{\alpha\beta}}{\partial \xi^\lambda} - \left(N^{\alpha\beta}\frac{\partial u_\alpha}{\partial \xi^\lambda}\right)_{,\beta} - \left(M^{\alpha\beta}\frac{\partial \phi_\beta}{\partial \xi^\lambda}\right)_{,\alpha} \\ & - \left(M^{\alpha\beta}_{,\alpha}\frac{\partial w}{\partial \xi^\lambda}\right)_{,\beta} - \frac{1}{2}\left[\frac{\partial u_\beta}{\partial \xi^\lambda}(b_\gamma{}^\beta M^{\alpha\gamma} - b_\gamma{}^\alpha M^{\beta\gamma})\right]_{,\alpha}\Big\}\,\mathrm{d}A. \end{aligned} \tag{A1}$$

Next, we make use of the strain-displacement relations (2), (3) and eqn (11) which expresses the fact the circular cylinder has a covariantly constant curvature tensor $b_\beta{}^\alpha$ and obtain

$$\begin{aligned} J_\lambda = \int_S \Big\{ & N^{\alpha\beta}\left(\frac{\partial u_{\alpha,\beta}}{\partial \xi^\lambda} + b_{\alpha\beta}\frac{\partial w}{\partial \xi^\lambda}\right) + M^{\alpha\beta}\left(-\frac{\partial w_{,\beta\alpha}}{\partial \xi^\lambda} + b_{\alpha\delta}\frac{\partial u^\delta{}_{,\beta}}{\partial \xi^\lambda}\right) + \frac{1}{2}M^{\alpha\beta}b_\alpha{}^\gamma\left(\frac{\partial u_{\gamma,\beta}}{\partial \xi^\lambda} - \frac{\partial u_{\beta,\gamma}}{\partial \xi^\lambda}\right) - N^{\alpha\beta}\frac{\partial u_{\alpha,\beta}}{\partial \xi^\lambda} - M^{\alpha\beta}\frac{\partial \phi_{\beta,\alpha}}{\partial \xi^\lambda} \\ & - M^{\alpha\beta}_{,\alpha}\frac{\partial w_{,\beta}}{\partial \xi^\lambda} - \frac{1}{2}\frac{\partial u_{\beta,\alpha}}{\partial \xi^\lambda}(b_\gamma{}^\beta M^{\alpha\gamma} - b_\gamma{}^\alpha M^{\beta\gamma}) - N^{\alpha\beta}_{,\alpha}\frac{\partial u_\alpha}{\partial \xi^\lambda} - M^{\alpha\beta}_{,\alpha}\frac{\partial \phi_\beta}{\partial \xi^\lambda} - M^{\alpha\beta}_{,\alpha\beta}\frac{\partial w}{\partial \xi^\lambda} - \frac{1}{2}\frac{\partial u_\beta}{\partial \xi^\lambda}(b_\gamma{}^\beta M^{\alpha\gamma} - b_\gamma{}^\alpha M^{\beta\gamma})_{,\alpha}\Big\}\,\mathrm{d}A. \end{aligned} \tag{A2}$$

After canceling certain terms performing some algebraic simplifications, we finally have

$$J_\lambda = \int_S \left\{ -\frac{\partial u_\delta}{\partial \xi^\lambda} \left[N^{\delta\beta}_{\cdot\beta} + b_\beta{}^\delta M^{\alpha\beta}_{\cdot\alpha} + \frac{1}{2}(b_\gamma{}^\delta M^{\alpha\gamma} - b_\gamma{}^\alpha M^{\delta\gamma})_{,\alpha} \right] - \frac{\partial w}{\partial \xi^\lambda} [M^{\alpha\beta}_{\cdot\alpha\beta} - b_{\alpha\beta} N^{\alpha\beta}] \right\} \mathrm{d}A \tag{A3}$$

and J_λ vanishes because of the equilibrium eqns (4).

For axisymmetric shells, $\sqrt{(g)} = R(z)\sqrt{(1 + R'(z)^2)}$ is a function of $\xi^1 = z$ only and hence $(\partial g / \partial \xi^2) = 0$. Also in the same lines-of-curvature coordinates $(\partial b_\beta{}^\alpha / \partial \xi^2) = 0$ so that the same proof above can be used in this case for $\lambda = 2$.

APPENDIX 2

Path independence for a nonlinear membrane cylindrical shell

For nonlinear membrane theory, the equilibrium equations when no pressures loads are present are [13],

$$[(g_{\gamma\beta} + d_{\gamma\beta}) n^{\alpha\beta}]_{,\alpha} - b_{\gamma\alpha} \phi_\beta n^{\alpha\beta} = 0 \tag{A4}$$

$$-(\phi_\beta n^{\alpha\beta})_{,\alpha} - b_\alpha{}^\gamma (g_{\gamma\beta} + d_{\gamma\beta}) n^{\alpha\beta} = 0. \tag{A5}$$

The above equations are exact and are derivable from the Principle of Virtual Work applied to the deformed shell.

Since the integrals in (28) are referred to the undeformed geometry of the circular cylinder, we can again apply the Divergence Theorem (10) with $\sqrt{(g)} = R$, radius of the underformed cylinder. This gives

$$J_\lambda = \int_S \left\{ n^{\alpha\beta} \frac{\partial \hat{E}_{\alpha\beta}}{\partial \xi^\lambda} - [(\delta_\beta{}^\gamma + d^\gamma_{\cdot\beta}) n^{\alpha\beta}]_{,\alpha} \frac{\partial u_\gamma}{\partial \xi^\lambda} - (\delta_\beta{}^\gamma + d^\gamma_{\cdot\beta}) n^{\alpha\beta} \times \frac{\partial u_{\gamma,\alpha}}{\partial \xi^\lambda} + \phi_\beta n^{\alpha\beta} \frac{\partial w_{,\alpha}}{\partial \xi^\lambda} + (\phi_\beta n^{\alpha\beta})_{,\alpha} \frac{\partial w}{\partial \xi^\lambda} \right\} \mathrm{d}A. \tag{A6}$$

Now using the nonlinear strain-displacement relation and applying the equilibrium equations (A6), (A7) give, after grouping certain terms together,

$$J_\lambda = \int_S \left\{ n^{\alpha\beta} \left(\frac{\partial E_{\alpha\beta}}{\partial \xi^\lambda} + d_{\gamma\beta} \frac{\partial d^\gamma_{\cdot\alpha}}{\partial \xi^\lambda} + \phi_\alpha \frac{\partial \phi_\beta}{\partial \xi^\lambda} \right) - \left[n^{\alpha\beta} \phi_\beta \frac{\partial \phi_\alpha}{\partial \xi^\lambda} + n^{\alpha\beta} (\delta_\beta{}^\gamma + d^\gamma_{\cdot\beta}) \left(\frac{\partial u_{\gamma,\alpha}}{\partial \xi^\lambda} + b_{\gamma\alpha} \frac{\partial w}{\partial \xi^\lambda} \right) \right] \right\} \mathrm{d}A. \tag{A7}$$

Using the symmetry of $n^{\alpha\beta}$ and expressing $E_{\alpha\beta}$, $d_{\alpha\beta}$ in terms of the displacements provide the result that J_λ vanishes for all closed paths C.